JOURNAL OF MULTIVARIATE ANALYSIS 7, 535–550 (1977)

On the Square Root of a Positive $B(\mathscr{X}, \mathscr{X}^*)$-Valued Function

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Communicated by M. M. Rao

Let $(\Omega, \mathfrak{B}, \mu)$ be a measure space, $\mathscr{X}$ a separable Banach space, and $\mathscr{X}^*$ the space of all bounded conjugate linear functionals on $\mathscr{X}$. Let f be a weak* summable positive $B(\mathscr{X}, \mathscr{X}^*)$-valued function defined on Ω. The existence of a separable Hilbert space $\mathscr{K}$, a weakly measurable $B(\mathscr{X}, \mathscr{K})$-valued function Q satisfying the relation $Q^*(\omega)Q(\omega) = f(\omega)$ is proved. This result is used to define the Hilbert space $L_{2,f}$ of square integrable operator-valued functions with respect to f. It is shown that for $B^+(\mathscr{X}, \mathscr{X}^*)$-valued measures, the concepts of weak*, weak, and strong countable additivity are all the same. Connections with stochastic processes are explained.

1. INTRODUCTION

Let $(\Omega, \mathfrak{B}, \mu)$ be a measure space and f a weak* measurable function defined on $(\Omega, \mathfrak{B}, \mu)$ whose values are nonnegative operators from a separable Banach space $\mathscr{X}$ into $\mathscr{X}^*$ the space of all bounded conjugate linear functionals on $\mathscr{X}$. The main purpose of this paper is to prove a splitting of f in the form $f = Q^*Q$, where Q is a weakly measurable operator-valued function whose values are bounded linear operators from $\mathscr{X}$ into some auxiliary Hilbert space $\mathscr{K}$. The measurable square root Q has application in the theory of Banach space-valued stationary processes, and plays an important role in the study of analytic factorization of the density of such processes. This can be seen in [17], where a similar splitting if obtained. In Remark 4.4 further discussion on the application of the square root to stochastic processes is included. In our present paper f is defined over any space Ω where in [17] f was defined on the interval $[0, 2\pi]$ which is endowed with algebraic and topological structures. To prove the splitting of f in [17] we appealed to a theorem on the existence of an auxiliary Hilbert space, due to Cŏbanjan [2] who strongly makes use of the structure of $[0, 2\pi]$. In this paper, using operator-values positive definite kernels, the existence of such

Received October 27, 1976; revised March 29, 1977.

AMS(MOS) 1970 subject classification: Primary 60G10, 46G99; Secondary 47B99.

Key words and phrases: Banach space, positive operator-valued functions, operator-valued positive definite kernels, square root, factorability, stationary processes.

ISSN 0047-259X

an auxiliary Hilbert space is proved for an arbitrary Ω. Then by the help of a certain result of Helson on factorization of infinite dimensional matrix-valued functions [7], measurability of Q is established. We remark that Helson's factorization result is proved for matrix-valued functions defined on $[0, 2\pi]$, but an inspection of his proof shows that it remains valid for any space Ω.

It is well known that the notions of weak and strong countable additivity for the Banach-space-valued measures are the same. The proof of this assertion is lengthy as is seen in [8]. Let M be a nonnegative operator-valued set function and let x be a vector in $\mathscr{X}$. We prove that for the vector-valued set function $M(\cdot)x$, these concepts are even equivalent to weak* countable additivity (see Theorem 2.7). We also prove a similar result concerning weak* and strong measurability of nonnegative operator-valued functions (see Remark 3.3).

In Section 2 we introduce some notations and state preliminary results from the theory of operator-values reproducing kernels as needed in later sections. In Section 3 a general factorization theorem for nonnegative $\mathscr{X}$ to $\mathscr{X}^*$ operator-valued measures is stated. Using this later result we establish the existence of a weakly measurable square root Q. We also prove the equivalence of weak*, weak, and strong countable additivity of $M(\cdot)x$ for any $\mathscr{X}$ to $\mathscr{X}^*$ nonnegative operator-valued measure M. In Section 4 we prove the completeness of the class of operator-valued functions which are square summable with respect to $fd\mu$, thereby extending some earher results of [13, 14].

2. Preliminary Results

Let $\mathscr{K}$ be a separable Hilbert space and let $L_2(\mathscr{K})$ denote the Hilbert space of all functions defined over a measure space $(\Omega, \mathfrak{B}, \mu)$ with values in $\mathscr{K}$, that are weakly measurable and have square summable norm with respect to (w.r.t.) measure μ. The $L_2(\mathscr{K})$ inner product of two functions g_1 and g_2 is given by $(g_1, g_2) = \int_\Omega (g_1(\omega), g_2(\omega))\, d_\mu(\omega)$. Since $\mathscr{K}$ is separable the notions of weak measurability and strong measurability coincide. Thus, $L_2(\mathscr{K})$ consists of strongly measurable functions.

Let $\mathscr{X}$ be a Banach space. $\mathscr{X}^*$ will denote the space of all bounded conjugate linear functionals on $\mathscr{X}$. For any two Banach spaces $\mathscr{X}$ and $\mathscr{Y}$, $B(\mathscr{X}, \mathscr{Y})$ denotes the class of all bounded linear operators on $\mathscr{X}$ into $\mathscr{Y}$. We should mention that the distinction between the adjoint $\mathscr{X}^*$ and the dual $\mathscr{X}'$ (the space of all bounded linear functionals on $\mathscr{X}$) of a Banach space $\mathscr{X}$ is important here,

$$\mathscr{X}^* = \{\overline{f(\cdot)}: f(\cdot) \in \mathscr{X}'\}.$$

It follows that if $T \in B(\mathscr{X}, \mathscr{X}^*)$, then $(T(\alpha x))(\beta y) = \bar{\beta}\alpha(Tx)(y)$. For T in $B(\mathscr{X}, \mathscr{Y})$ the adjoint T^* of T is the bounded operator from $\mathscr{Y}^*$ into $\mathscr{X}^*$ defined by

$$T^*(y^*) = y^* \circ T, \qquad y^* \in \mathscr{Y}^*.$$

If $\mathscr{H}$ is a Hilbert space then $\mathscr{H}^*$ is isometrically isomorphic to $\mathscr{H}$ under the map $h \leftrightarrow (h, \cdot)$, $h \in \mathscr{H}$. In this paper we identify $\mathscr{H}^*$ and $\mathscr{H}$. Consequently, the adjoint T^* of an operator T in $B(\mathscr{X}, \mathscr{H})$ belongs to $B(\mathscr{H}, \mathscr{X}^*)$. With this identification and this definition of adjoint, it readily follows that for $S, T \in B(\mathscr{X}, \mathscr{H})$,

$$(T^*Sx)(y) = (Sx)(Ty) = (Sx, Ty)_{\mathscr{H}}, \qquad \text{for all} \quad x, y \in \mathscr{X}.$$

In this section we will state some known results on the theory of operator-valued positive definite kernels which are used in later sections. The theory of positive definite kernels and their associated reproducing kernel Hilbert spaces has been developed and studied extensively in the literature [1, 24]. Positive definite kernels arise naturally in the study of second-order stochastic processes [9, 11, 12, 22]. In fact the covariance operator of any stochastic process is a positive definite kernel. In particular the covariance operator of a Gaussian process taking value in a separable Banach space is the well-known Wiener kernel, and its reproducing kernel consists of absolutely continuous functions whose derivatives are square summable. The exact form of this positive definite kernel and its reproducing kernel space is given in [11]. Another example of positive definite kernel (as one can see in Theorem 2.8) is $K(A, B) = M(A \cap B)$, where M is a nonnegative $B(\mathscr{X}, \mathscr{X}^*)$-valued function and $A, B \in \mathfrak{B}$.

2.1. Definition. Let A be in $B(\mathscr{X}, \mathscr{X}^*)$. A is said to be Hermitian if for all $x_1, x_2 \in \mathscr{X}$, $(Ax_1)(x_2) = \overline{(Ax_2)(x_1)}$. A is said to be nonnegative if A is Hermitian and for all $x \in \mathscr{X}$, $(Ax)(x) \geqslant 0$. $B^+(\mathscr{X}, \mathscr{X}^*)$ denotes the class of all nonnegative elements of $B(\mathscr{X}, \mathscr{X}^*)$.

2.2. Definition. Let Λ be a given set and let F be a Hilbert space of $\mathscr{X}^*$-valued functions defined on Λ. The inner product and norm in F are denoted by $(\cdot, \cdot)_F$ and $|\cdot|_F$, respectively. A reproducing kernel (r.k.) for F is a function $K(\cdot, \cdot) \cdot \Lambda \times \Lambda \to B(\mathscr{X}, \mathscr{X}^*)$ such that

(i) for all $\lambda \in \Lambda$ and all $x \in \mathscr{X}$, $K(\lambda, \cdot)x \in F$, and

(ii) for all $f \in F$, $\lambda \in \Lambda$, $x \in \mathscr{X}$ we have

$$f(\lambda)x = (f(\cdot), K(\lambda, \cdot)x)_F .$$

It is easy to see that when a r.k. exists for a Hilbert space F, it is unique.

2.3. Definition. A kernel $K(\cdot, \cdot): \Lambda \times \Lambda \to B(\mathscr{X}, \mathscr{X}^*)$ is said to be a positive definite kernel (p.d.k.) if for any choice of $\lambda_1, \ldots, \lambda_n \in \Lambda$; $x_1, \ldots, x_n \in \mathscr{X}$ we have

$$\sum_{j=1}^{n} \sum_{i=1}^{n} (K(\lambda_i, \lambda_j)x_i)(x_j) \geqslant 0. \tag{2.1}$$

Note that when $\mathscr{X}$ is a complex Banach space (1) implies that

$$(K(\lambda_1, \lambda_2)x_1)(x_2) = \overline{(K(\lambda_2, \lambda_1)x_2)(x_1)}. \tag{2.2}$$

But when $\mathscr{X}$ is a real Banach space in addition to (1.1) we also assume

$$(K(\lambda_1, \lambda_2)\, x_1)(x_2) = (K(\lambda_2, \lambda_1)\, x_2)(x_1), \tag{2.3}$$

which amounts to (2.2) in the case of complex Banach spaces.

2.4. LEMMA. *Let $K(\cdot, \cdot)$ be a r.k. for the Hilbert space F over $\Lambda \times \Lambda$. Then we have*

$$K(\lambda_1, \lambda_2) = T^*(\lambda_2)\, T(\lambda_1); \qquad \lambda_1, \lambda_2 \in \Lambda,$$

where $T(\cdot)$ is a $B(\mathscr{X}, F)$-valued fubction on Λ. As a result $K(\cdot, \cdot)$ is a p.d.k.

Proof. Define $T(\lambda)x = K(\lambda, \cdot)x$, $\lambda \in \Lambda$, $x \in \mathscr{X}$. It easily follows that $T(\lambda) \in B(\mathscr{X}, F)$. Now for each $x, y \in \mathscr{X}$ we have

$$\begin{aligned}(T^*(\lambda_2)T(\lambda_1)x)(y) &= (T(\lambda_1)x, T(\lambda_2)y)_F \\ &= (K(\lambda_1, \cdot)x, K(\lambda_2, \cdot)y)_F \\ &= (K(\lambda_1, \lambda_2)x)(y).\end{aligned}$$

Hence

$$K(\lambda_1, \lambda_2) = T^*(\lambda_2)\, T(\lambda_1).$$

2.5. LEMMA. *To every $B(\mathscr{X}, \mathscr{X}^*)$-valued p.d.k. $K(\cdot, \cdot)$ on $\Lambda \times \Lambda$ there corresponds exactly one class of $\mathscr{X}^*$-valued functions F on Λ with an inner product forming a Hilbert space and admitting $K(\cdot, \cdot)$ as its r.k.*

Proof. We let F' be the linear space of all $\mathscr{X}^*$-valued functions on Λ which are of the form

$$f(\lambda) = \sum_{i=1}^{n} K(\lambda_i, \lambda)x_i; \qquad \lambda_i \in \Lambda, \quad x_i \in \mathscr{X}.$$

For any two functions $f(\lambda) = \sum_{i=1}^{n} K(\lambda_1, \lambda)\, x_i$ and $g(\lambda) = \sum_{j=1}^{m} K(\lambda_j', \lambda)\, x_j'$ in F' we define $(f, g)_{F'}$ by

$$(f, g)_{F'} = \sum_{j=1}^{m} \sum_{i=1}^{n} (K(\lambda_i, \lambda_j')x_i)(x_j').$$

Now $(\cdot, \cdot)_{F'}$ is independent of the choice of representations for f and g. Because if $\sum_{i=1}^{m} K(\lambda_i, \cdot)x_i$ and $\sum_{j=1}^{n} K(\mu_j, \cdot)y_j$ are two representations of f and $\sum_{k=1}^{l} K(\nu_k, \cdot)\, z_k$ is a representation for g, then

$$\sum_{i=1}^{m} (K(\lambda_i, \lambda)x_i)(x) = \sum_{j=1}^{n} (K(\mu_j, \lambda)y_j)(x), \qquad \lambda \in \Lambda, \quad x \in \mathscr{X},$$

and hence

$$\left(\sum_{i=1}^{m} K(\lambda_i, \cdot)x_i, \sum_{k=1}^{l} K(\nu_k, \cdot)z_k\right)_{F'}$$

$$= \sum_{i=1}^{m} \sum_{k=1}^{l} (K(\lambda_i, \nu_k)x_i)(z_k)$$

$$= \sum_{k=1}^{l} \sum_{j=1}^{n} (K(\mu_j, \nu_k)y_j)(z_k) = \left(\sum_{j=1}^{n} \sum_{k=1}^{l} (K(\mu_j, \nu_k)y_j)(z_k)\right)$$

$$= \left(\sum_{j=1}^{n} K(\mu_j, \cdot)y_j, \sum_{k=1}^{l} K(\nu_k, \cdot)z_k\right)_{F'}.$$

Let $f = f_1 = f_2$ and $g = g_1 = g_2$ where $f_1, f_2, g_1, g_2 \in F'$; then by the above argument we have

$$(f_1, g_1)_{F'} = (f_2, g_1)_{F'} = \overline{(g_1, f_2)_{F'}} = \overline{(g_2, f_2)_{F'}}$$
$$= (f_2, g_2)_{F'}.$$

For each $f \in F'$, $x \in \mathscr{X}$, and $\lambda \in \Lambda$, one can see that

$$\begin{aligned} f(\lambda)x &= (f(\cdot), K(\lambda, \cdot)x)_{F'}, \\ |f(\lambda)|^2_{\mathscr{X}^*} &\leqslant |f(\cdot)|^2_{F'} \, \| K(\lambda, \lambda)\|; \end{aligned} \tag{A}$$

one can now easily verify that the function $(\cdot, \cdot)_{F'}$ defines an inner product on F'. Now let $\{f_n\}$ be a Cauchy sequence in F' w r.t. $|\cdot|_{F'}$ norm, then by (A) for each λ, $f_n(\lambda)$ converges to some element, say $f(\lambda)$, in $\mathscr{X}^*$. Let F denote the class of all $\mathscr{X}^*$-valued functions f obtained this way. Clearly F is a linear manifold of $\mathscr{X}^*$-valued functions containing F'.

For any f and g in F we define $(f, g)_F$ by $(f, g)_F = \lim_{n\to\infty} (f_n, g_n)$, where $\{f_n\}$ and $\{g_n\}$ are Cauchy sequences in F' such that $f_n(\lambda) \to f(\lambda)$ and $g_n(\lambda) \to g(\lambda)$ for each $\lambda \in \Lambda$. One can see that $(f, g)_F$ is independent of the choice of Cauchy sequences determining f and g and defines an inner product on F which makes it a Hilbert space. One can also verify that

(1) if $\{f_n\}$ is a Cauchy sequence in F' determining f then $|f_n - f| \to 0$, as $n \to \infty$,

(2) $F, (\cdot, \cdot)_F$ is a Hilbert space, and

(3) $K(\cdot, \cdot)$ is the r.k. for F, and (A) holds for the functions in F as well.

2.6. THEOREM. *Let $K(\cdot, \cdot)$ be a $B(\mathscr{X}, \mathscr{X}^*)$-valued p.d.k. on $\Lambda \times \Lambda$. Then:*

(a) *There exists a Hilbert space* $\mathscr{K}$ *and a* $B(\mathscr{X}, \mathscr{K})$*-valued function* $T(\cdot)$ *on* Λ *such that for all* $\lambda_1, \lambda_2 \in \Lambda$ *we have*

$$K(\lambda_1, \lambda_2) = T^*(\lambda_2)\, T(\lambda_1).$$

(b) *The Hilbert space* $\mathscr{K}$ *and the* $B(\mathscr{X}, \mathscr{K})$*-valued function* $T(\cdot)$ *on* Λ *are unique, in the sense that if* $\mathscr{H}$ *and* $\mathscr{K}$ *are two Hilbert spaces and* $S(\cdot)$, $T(\cdot)$ *are, respectively,* $B(\mathscr{X}, \mathscr{H})$-, $B(\mathscr{X}, \mathscr{K})$*-valued functions on* Λ *such that*

$$S^*(\lambda_2)\, S(\lambda_1) = T^*(\lambda_2)\, T(\lambda_1).$$

Then there exists a unitary U *from* $\mathfrak{S}(s(\lambda)\mathscr{X}, \lambda \in \Lambda)$ *onto* $\mathfrak{S}(T(\lambda)\mathscr{X}, \lambda \in \Lambda)$ *such that*

$$T(\lambda) = US(\lambda)$$

{*for any subset* A *of a Hilbert space* $\mathscr{H}$, $\mathfrak{S}(A) =$ *the closed subspace spanned by* A}.

Proof. Part (a) is a consequence of Lemmas 2.4 and 2.5. Part (b) can be easily proved.

Let $\mathfrak{B}$ be a $\mathfrak{S}$-algebra of subsets of a space Ω and let M be a $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function on $\mathfrak{B}$. Let B_i denote an arbitrary sequence of mutually exclusive sets in $\mathfrak{B}$. We say that M is strongly countably additive if for all $x \in \mathscr{X}$,

$$\left\| M\left(\bigcup_{i=1}^{\infty} B_i\right) x - \sum_{i=1}^{n} M(B_i)x \right\|_{\mathscr{X}^*} \to 0 \qquad \text{as} \quad n \to \infty;$$

weakly countably additive if for each $x \in \mathscr{X}$ and $x^{**} \in \mathscr{X}^{**}$;

$$x^{**}\left(M\left(\bigcup_{i=1}^{\infty} B_i\right) x - \sum_{i=1}^{n} M(B_i)x\right) \to 0 \qquad \text{as} \quad n \to \infty;$$

and weak* countably additive if for each x and y in $\mathscr{X}$ we have

$$\left(\left(M\left(\bigcup_{i=1}^{\infty} B_i\right) - \sum_{i=1}^{n} M(B_i)\right) x\right)(y) \to 0 \qquad \text{as} \quad n \to \infty.$$

In the next theorem we show that for such measures all these three kinds of countable additives are the same.

2.7. Theorem. *Let* $\mathfrak{B}$ *be a* $\mathfrak{S}$*-algebra of subsets of a space* Ω *and let* M *be a* $B(\mathscr{X}, \mathscr{X}^*)$*-valued function on* $\mathfrak{B}$*. Then the above three notions of countable additivity are equivalent.*

Proof. It is obvious that strong countable additivity $\Rightarrow$ weak countable additivity $\Rightarrow$ weak* countable additivity.

It is enough to show that weak* countable additivity $\Rightarrow$ strong countable additivity, and this is the subject of part (d) of the following theorem.

2.8. THEOREM. *Let $\mathfrak{B}$ be a $\mathfrak{S}$-algebra of subsets of Ω, and let M be a weak* countably additive $B^+(\mathscr{X}, \mathscr{X}^*)$-valued measure on $\mathfrak{B}$. Define the kernel $k(\cdot, \cdot)$ on $\mathfrak{B} \times \mathfrak{B}$ by:*

$$K(A, B) = M(A \cap B); \qquad A, B \in \mathfrak{B}.$$

Then

(a) *$K(\cdot, \cdot)$ is a $B(\mathscr{X}, \mathscr{X}^*)$-valued p.d.k. on $\mathfrak{B} \times \mathfrak{B}$.*

(b) *There exists a Hilbert space $\mathscr{K}$ and a $B(\mathscr{X}, \mathscr{K})$-valued function $T(\cdot)$ on $\mathfrak{B}$ such that*

$$K(A, B) = T^*(B)T(A); \qquad A, B \in \mathfrak{B}.$$

For each x in $\mathscr{X}$, the $\mathscr{K}$-valued function $T(\cdot)x$ is countably additive and orthogonally scattered {in the language of [15, 16], $T(\cdot)$ is countably additive quasi-isometric}.

(c) *For each $B \in \mathfrak{B}$ we define $E(B)$ by $E(B) =$ orthogonal projection of $\mathscr{M}_T(\Omega) = \mathfrak{S}\{T(\Delta)\mathscr{X};\ \Delta \in \mathfrak{B}\}$ onto $\mathscr{M}_T(B) = \mathfrak{S}\{T(\Delta)\mathscr{X};\ \Delta \in \mathfrak{B}, \Delta \subset B\}$. Then E is a strongly countably additive spectral measure for the Hilbert space $\mathscr{M}_T(\Omega)$, and, moreover,*

$$M(B) = T^*(\Omega)\, E(B)\, T(\Omega), \qquad B \in \mathfrak{B}.$$

(d) *M is strongly countably additive.*

Proof. (a) Weak* countable additivity of M implies its finite additivity in any sense. But under the finite additivity of $M(\cdot)$ it is shown in [16] that $K(\cdot, \cdot)$ is a p.d.k.

(b) The first statement follows from (a) and Theorem 2.6. The last statement is obvious. It remains to show that T is strongly countably additive. Let $B_1, B_2, \ldots, B_n$ be mutually exclusive subsets in $\mathfrak{B}$; C any arbitrary subset in $\mathfrak{B}$; and $x, y \in \mathscr{X}$. *Then*

$$\begin{aligned}
\left(\sum_{i=1}^{n} T(B_i)x, T(C)y\right) &= \sum_{i=1}^{n} (T(B_i)x, T(C)y) \\
&= \sum_{i=1}^{n} (M(B_i \cap C)x)(y) = \left(M\left(\bigcup_{i=1}^{n} (B_i \cap C)\right) x\right)(y) \\
&= \left(M\left(\left(\bigcup_{i=1}^{n} B_i\right) \cap C\right) x\right)(y) \\
&= \left(T\left(\bigcup_{i=1}^{n} B_i\right) x, T(C)y\right).
\end{aligned}$$

Thus, for any linear combination $c = \sum_{j=1}^{m} \alpha_j T(C_j)\, y_j$,

$$\left(\sum_{i=1}^{n} T(B_i)x, c\right) = \left(\sum_{i=1}^{n} T(B_i)x, \sum_{j=1}^{m} \alpha_j T(C_j)y_j\right)$$

$$= \sum_{j=1}^{m} \bar{\alpha}_j \left(\sum_{i=1}^{n} T(B_i)x, T(C_j)y_j\right)$$

$$= \sum_{j=1}^{m} \bar{\alpha}_j \left(T\left(\bigcup_{i=1}^{n} B_i\right) x, T(C_j)y_j\right)$$

$$= \left(T\left(\bigcup_{i=1}^{n} B_i\right) x, \sum_{j=1}^{m} \alpha_j T(C_j)y_i\right)$$

$$= \left(T\left(\bigcup_{i=1}^{n} B_i\right) x, c\right).$$

Hence, using the continuity of the inner product, we see that the last equality is valid for any c in the Hilbert space $H = \mathfrak{S}\{T(B)x, B \in \mathfrak{B}\}$. Therefore $T(\bigcup_{i=1}^{n} B_i)x = \sum_{i=1}^{n} T(B_i)x$ or equivalently $T(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} T(B_i)$, i.e., T is finitely additive. Let $A, B \in \mathfrak{B}$ and $x \in \mathscr{X}$; then it is clear that

$$| T(A)x - T(B)x |_{\mathscr{X}}^2 = ((K(A, A) + K(B, B) - K(A, B) - K(B, A))x)(x)$$
$$= ((M(A) + M(B) - 2M(A \cap B))x)(x).$$

Now if $\{B_i\}$ is a sequence of mutually exclusive elements of $\mathfrak{B}$ and $x \in \mathscr{X}$, then

$$\left| T\left(\bigcup_{i=1}^{\infty} B_i\right) x - T\left(\bigcup_{i=1}^{n} B_i\right) x \right|_{\mathscr{X}}^2$$
$$= \left(\left(M\left(\bigcup_{i=1}^{\infty} B_i\right) + M\left(\bigcup_{i=1}^{n} B_i\right) - 2M\left(\bigcup_{i=1}^{n} B_i\right)\right) x\right)(x).$$

Now using the finite additivity of T and M,

$$\left| T\left(\bigcup_{i=1}^{\infty} B_i\right) x - \sum_{i=1}^{n} T(B_i)x \right|_{\mathscr{X}}^2 = \left(M\left(\bigcup_{1}^{\infty} B_i\right) x\right) x - \sum_{i=1}^{n} (M(B_i)x)(x).$$

Taking limits as $n \to \infty$ and using the weak* countably additivity of M we get

$$\left| T\left(\bigcup_{i=1}^{\infty} B_i\right) x - \sum_{i=1}^{n} T(B_i)x \right|_{\mathscr{X}} \to 0, \qquad \text{as} \quad n \to \infty,$$

which means the strong countable additivity for $T(\cdot)x$, $x \in \mathscr{X}$.

(c) Having the strong countable additivity of T, the proof of (c) may be completed in exactly the same way as appeared in [16, Theorem 5.9 and Appendix H]. In [16] the operator $T(\cdot)$ is assumed to be from a Hilbert space $\mathscr{H}$ into a Hilbert space $\mathscr{H}$, and in our case $T(\cdot)$ maps a Banach space $\mathscr{X}$ into a Hilbert space $\mathscr{K}$. We point out that the range of T, which is a Hilbert space in either case, is the crucial factor in proving the result.

(d) Let $\{B_i\}$ be a sequence of mutually exclusive sets in $\mathfrak{B}$, and $x \in \mathscr{X}$, then using strong countable additivity of E, and boundedness of T^*, we get

$$M\left(\bigcup_{i=1}^{\infty} B_i\right) x = \left(T^*(\Omega) E\left(\bigcup_{i=1}^{\infty} B_i\right) T(\Omega) x\right)$$

$$= T^*(\Omega)\left[\lim_{n\to\infty} \sum_{i=1}^{n} E(B_i)\, T(\Omega) x\right]$$

$$= \lim_{n\to\infty} T^*(\Omega)\left[\sum_{i=1}^{n} E(B_i)\, T(\Omega) x\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} T^*(\Omega)[E(B_i)\, T(\Omega) x] = \sum_{i=1}^{\infty} T^*(\Omega)\, E(B_i)\, T(\Omega) x.$$

Thus, $M(\bigcup_{i=1}^{\infty} B_i)x = \sum_{i=1}^{\infty} M(B_i)x$, which means strong countable additivity of M.

We remark that the equivalence of weak and strong countable additivity, as defined in the paragraph preceding Theorem 2.7, also follows from [8, Theorem 3.6.2].

Also Theorem 2.8 gives an explicit form of a result due to Naimark for the Hilbert spaces [21].

Now that we have seen the equivalence of these countable additivities, we would omit prefixes weak*, weak, and strong; and use the phrase countable additivity (c.a.) for all of these.

3. Measurable Square Roots

In the rest of this paper we will take $\mathscr{X}$ to be a separable Banach space. A $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function $f(\cdot)$ over $(\Omega, \mathfrak{B}, \mu)$ is said to be weak* summable if, for each $x, y \in \mathscr{X}$, the scalar-valued function $(f(\cdot)x)(y)$ is summable w.r.t. μ.

In this section we will show that any weak* summable $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function $f(\cdot)$ over $(\Omega, \mathfrak{B}, \mu)$ admits a strongly measurable square root $Q(\cdot)$ in the following sense:

$$f(\omega) = Q^*(\omega)\, Q(\omega) \qquad \text{almost every } \omega.$$

(For definition of various kinds of measurability for operator-valued functions see ([8, Definition 3.5.5].)

Let $f(\cdot)$ be a weak* summable $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function over $(\Omega, \mathfrak{B}, \mu)$. For each fixed $B \in \mathfrak{B}$ we define the bilinear form

$$G_B(x, y) = \int_B (f(\omega)x)(y)\, d\mu(\omega); \qquad x, y \in \mathscr{X}.$$

We want to show this is a bounded bilinear form. For each $y \in \mathscr{X}$ define a linear map $T_{B,y}: \mathscr{X} \to L_1(\Omega, \mathfrak{B}, \mu)$ by $T_{B,y}(x) = I_B(f(\cdot)x)(y)$. The domain of $T_{B,y}$ consists of all of $\mathscr{X}$. It is easy to see that $T_{B,y}$ is closed. Hence $T_{B,y}$ is bounded, i.e., there exists a constant $M_{B,y} > 0$ such that $| T_{B,y}(x)|_{L_1} \leqslant M_{B,y} \| x \|$. Therefore $| G_B(x, y)| \leqslant | T_{B,y}(x)|_{L_1} \leqslant M_{B,y} \| x \|$; $x, y \in \mathscr{X}$. So

$$| G_B(x, y)| \leqslant M_{B,y}; \qquad \| x \| = 1, \quad y \in \mathscr{X}. \tag{3.1}$$

We note that (3.1) implies that $G_B(\cdot, y)$ is a bounded linear functional on $\mathscr{X}$. Similarly we can show that for each $x \in \mathscr{X}$ $\overline{G_B(x, \cdot)}$ is a bounded linear functional on $\mathscr{X}$. So for each x, $\| x \| = 1$, by (3.1), we have $| \overline{G_B(x, y)}| \leqslant M_{B,y}$. Hence by the uniform boundedness principle we have

$$\sup_{\|x\|=1} [\sup_{\|y\|=1} | \overline{G_B(x, y)}|] < C_B < \infty.$$

Therefore,

$$| G_B(x, y)| \leqslant C_B \| x \| \| y \|; \qquad x, y \in \mathscr{X}.$$

Thus the bilinear form $G_B(x, y)$ is bounded, and hence the relation $(F(B)x)(y) = G_B(x, y)$ defines a nonnegative bounded linear operator $F(B)$ from $\mathscr{X}$ into $\mathscr{X}^*$. We define the integral $\int_B f\, d\mu$ to be this operator $F(B)$, and we will write

$$F(B) = \int_B f\, d\mu.$$

Clearly the function $F(\cdot)$ is c.a. $B^+(\mathscr{X}, \mathscr{X}^*)$-valued measure on $(\Omega, \mathfrak{B})$. From Theorem 2.8 we obtain the existence of a Hilbert space $\mathscr{K}$, an operator T in $B(\mathscr{X}, \mathscr{K})$, and a spectral measure $E(\cdot)$ for the Hilbert space $\mathscr{M}_T(\Omega) \subseteq \mathscr{K}$ such that

$$F(B) = T^*E(B)T, \qquad \text{for all} \quad B \in \mathfrak{B}.$$

In a manner similar to [17, Lemma 2.5], we prove the following lemma.

3.1. Lemma. *With the above notations the restriction of the spectral measure E to $\mathfrak{A} = \mathfrak{S}\{T\mathscr{X}\}$ is weak* absolutely continuous w.r.t μ, and it has a density $dE/d\mu = g$ in the weak* sense, i.e.,*

$$\frac{d(E(\cdot)h, k)}{d\mu} = g(\cdot)(h, k), \qquad \textit{for all } h, k \in \mathfrak{A}.$$

Where for each $\omega \in \Omega$, $g(\omega)$ is a bilinear (not necessarily bounded) form on $\mathfrak{A} \times \mathfrak{A}$. Moreover, for each $x, y \in \mathscr{X}$, we have

$$(f(\omega)x)(y) = g(\omega)(Tx, Ty), \qquad \text{almost every } \omega.$$

Proof. Let $h = Tx$ for $x \in \mathscr{X}$. Then for each $B \in \mathfrak{B}$ we have

$$\begin{aligned}(E(B)h, h) &= (E(B)\, Tx, Tx) = (T^*E(B)\, Tx)(x)\\ &= (F(B)x)(x) = G_B(x, x) = \int_B (f(\omega)x)(x)\, d\mu(\omega).\end{aligned}$$

Hence $(E(\cdot)h, h)$ is absolutely continuous w.r.t. μ, for each $h \in T(\mathscr{X})$. By [6, Sect. 66] we deduce that $(E(\cdot)h, h)$ is absolutely continuous w.r.t. μ, for each $h \in \mathfrak{A} = \mathfrak{S}\{Tx\}$. Therefore by the polarization theorem $(E(\cdot)h, k)$ is absolutely continuous w.r.t. μ. Since $(E(B)h, k)$ is a bilinear form so is its derivative $g(\omega)(h, k)$. Now for each $x, y \in \mathscr{X}$ we have

$$(f(\omega)x)(y) = \frac{d(E(\cdot)\, Tx, Ty)}{d\mu} = g(\omega)(Tx, Ty) \qquad \text{almost every } \omega.$$

3.2. THEOREM. *Let f be a weak* summable $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function defined on the measure space $(\Omega, \mathfrak{B}, \mu)$. Then there exists a separable Hilbert space $\mathscr{K}$, and is strongly measurable $B(\mathscr{X}, \mathscr{K})$-valued function Q over $(\Omega, \mathfrak{B}, \mu)$ such that*

$$f(\omega) = Q^*(\omega)\, Q(\omega), \qquad \textit{almost every } \omega.$$

The Hilbert space $\mathscr{K}$ and the factor Q are unique, in the following sense that, if $\mathscr{K}_1$, $\mathscr{K}_2$ are two such Hilbert spaces and Q_1, Q_2 are two such factors, then there exists a unitary map V on $\mathfrak{S}\{Q_1(\omega)\mathscr{X}, \omega \in \Omega\} \subseteq \mathscr{K}_1$ onto $\mathfrak{S}\{Q_2(\omega)\mathscr{X}, \omega \in \Omega\} \subseteq \mathscr{K}_2$ such that

$$Q_1(\omega) = VQ_2(\omega), \qquad \textit{almost every } \omega.$$

Proof. (a) By Lemma 3.1 there exists a Hilbert space $\mathfrak{A}$, an operator T in $B(\mathscr{X}, \mathfrak{A})$, and a function g on Ω with $\mathfrak{A} = \mathfrak{S}\{T\mathscr{X}\}$, such that for each $x, y \in \mathscr{X}$,

$$(f(\omega)x)(y) = g(\omega)(Tx, Ty) \qquad \text{almost every } \omega. \tag{3.2}$$

Let $\{x_i\}_{i=1}^\infty$ be a countable dense set in $\mathscr{X}$ and let $\{e_i\}_{i=1}^\infty$ be the Gram–Schmidt orthogonalization of $\{Tx_i\}_{i=1}^\infty$. Let

$$g_{ij}(\omega) = g(\omega)(e_i, e_j).$$

It is clear that $g_{ij}(\omega)$ defines a nonnegative matrix (not necessarily bounded). An inspection of the proof of a result due to Helson [7, p. 112] shows that his result remains valid when $[0, 2\pi]$ is replaced by Ω. With this modification, the above result of Helson can be applied to show the existence of a separable

Hilbert space $\mathscr{K}$, and a sequence $\{F_i\}_{i=1}^{\infty}$ in $L_2(\mathscr{K})$ such that $g_{ij}(\omega) = (F_i(\omega), F_j(\omega))_{\mathscr{K}}$. Following [17, p. 106], we obtain an operator A from the linear span (not closed) of $\{e_i\}_{i=1}^{\infty}$ into $L_2(\mathscr{K})$ by

$$A\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i F_i . \tag{3.3}$$

Using (3.2) it is easy to see

$$| ATx |_{\mathscr{K}}^2 = (f(\omega)x)(x), \qquad \text{for all } x \in \{x_i \colon 1 \leqslant i < \infty\}. \tag{3.4}$$

Since f is weak* summable, (3.4) shows that $AT\colon \mathscr{X} \to L_2(\mathscr{K})$ is defined for all x_i, $1 \leqslant i < \infty$. Now for almost every ω, we define $B(\omega)$ on $\{x_i, 1 \leqslant i < \infty\}$ by $B(\omega)x = (ATx)(\omega)$. It follows (see [17, p. 106]) that for almost every ω, $B(\omega)$ is bounded and hence can be extended to $\mathscr{X}$. We call this extension $Q(\omega)$. It is clear that

$$Q^*(\omega)Q(\omega) = f(\omega), \qquad \text{almost every } \omega.$$

Furthermore, (3.3) shows that Q is strongly measurable. The rest of the proof is easy and is omitted.

3.3. *Remark.* From the aplitting $f = Q^*Q$ and the strong measurability of Q it follows that any weak* summable $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function f is also strongly measurable.

3.4. Definition. The function Q in Theorem 3.2 is called a measurable square root for f.

4. The Space $L_{2,f}$

In this section we use the notion of square root to define the space of square integrable operator-valued functions w.r.t. $f d\mu$, and we show that this space becomes a Hilbert space under an appropriate inner product. Because of similarities which exist between this case and the corresponding results when $\mathscr{X}$ is a Hilbert space [13] we will summarize our results. The important tool here is the availability of a measurable square root which is given in Theorem 3.2. We remark that, as in [13, 14], to conclude the completeness of the space $L_{2,f}$, we also have to consider unbounded linear operators in defining the space $L_{2,f}$.

4.1. Definition. Let $\mathscr{K}$ and $\mathscr{H}$ be two separable Hilbert spaces. Then space $HS(\mathscr{K}, \mathscr{H})$ of all Hilbert–Schmidt operators on $\mathscr{K}$ into $\mathscr{H}$ is a separable Hilbert space under the inner product $(S, T) = \operatorname{trace}(ST^*)$. Since this is a separable Hilbert space we can consider $L_2(HS(\mathscr{K}, \mathscr{H}))$ [see the definition of

$L_2(\mathscr{K})$ at the beginning of Section 2] which is a Hilbert space under the inner product $(\Phi, \Psi) = \int_\Omega \operatorname{trace}(\Phi\Psi^*)\, d\mu$.

4.2. DEFINITION. Let f be a weak* measurable $B^+(\mathscr{X}, \mathscr{X}^*)$-valued function on the measure space $(\Omega, \mathfrak{B}, \mu)$, and let Q be a $B(\mathscr{X}, \mathscr{K})$-valued measurable square root for f. Let $\mathscr{H}$ be a separable Hilbert space. A function Φ on Ω, whose values are linear transformations (not necessarily bounded) from $\mathscr{X}^*$ into $\mathscr{H}$ is called square integrable w.r.t. $f\, d\mu$ if ΦQ^* belongs to $L_2(HS(\mathscr{K}, \mathscr{H}))$. We will denote by $L_{2,f}$ the family of all equivalent classes of such square integrable functions. In $L_{2,f}$ we define the inner product

$$((\Phi, \Psi)) = \int_\Omega \operatorname{trace}(\Phi Q^*)(\Psi Q^*)^*\, d\mu,$$

and identify Φ and Ψ if $((\Phi - \Psi, \Phi - \Psi)) = 0$. Define the map V on $L_{2,f}$ into $L_2(HS(\mathscr{K}, \mathscr{H}))$ by $V\Phi = \Phi Q^*$. Clearly V is an isometry from $L_{2,f}$ into $L_2(HS(\mathscr{K}, \mathscr{H}))$. In the next theorem the completeness of $L_{2,f}$ is stated. The proof can be completed by showing that range of V is a closed subspace of $L_2(HS(\mathscr{K}, \mathscr{H}))$.

Let $P(\omega)$ denote the orthogonal projection of $\mathscr{K}$ onto the orthogonal complement of the null space of $Q^*(\omega)$. Then $P(\omega)$ is measurable [14].

4.3. THEOREM. (a) $VL_{2,f} = \{\Phi \in L_2(HS(\mathscr{K}, \mathscr{H})) : \Phi P = \Phi\}$.

(b) *The space $L_{2,f}$ together with the inner product*

$$((\Phi, \Psi)) = \int_\Omega \operatorname{trace}(\Phi Q^*)(\Phi Q^*)^*\, d\mu$$

is a Hilbert space.

4.4. *Remark.* (a) Our original interest in the problems discussed in this paper stemmed from the study of Banach-space-valued stationary stochastic processes. In [10], Kolmogorov studied second-order stationary processes as curves in Hilbert spaces. He proved fundamental results on the theory of such processes. In his work he established the well-known isomorphism theorem between the time and spectral domain. The extension of Kolmogorov's work to multivariate processes was considered by several authors [25, 27]. In the study of multivariate stationary processes the Hilbert space $L_{2,M}$ of all square summable vector-valued functions with respect to a nonnegative (finite dimensional) matrix-valued measure M plays an important role. Infinite dimensional stationary processes has been studied by several authors, such as [5, 13, 19, 20, 23]. Here also the space $L_{2,M}$ is crucial in the analysis of stationary processes. The idea of Banach-space-valued stationary stochastic processes was introduced in [2] and subsequently some basic results concerning these processes were announced in [3, 4]. [English translations of several papers including these are avaiable at Michigan State University Mathematics Library (by A G. Miamee).] In [17, 18] the study of these processes was further pursued. The covariance functions of these processes are $B(\mathscr{X}, \mathscr{X}^*)$-valued positive definite kernels. This leads, via Bochner's theorem of such kernels, to the spectral distribution F and its density f, with values in $B^+(\mathscr{X}, \mathscr{X}^*)$. The usual properties of time domain of a process can be explained by the study of the density f and the corresponding space $L_{2,f}$. For instance it is shown in [18] that a process is regular if and only if its density f is factorable (for terminologies see [18]). So the problem of factorability of such densities in this connection becomes important. In order to study the factorability question and related problems in time domain of a stationary process, it appears that the availability of a measurable "square root" for f is crucial [17, 26]. We may add that when a process is indexed by integers the spectral density f is defined on $[0, 2\pi]$. But when the process is indexed by elements of some arbitrary locally compact Abelian group G, its spectral density f will be defined on its dual group $\hat{G}$. Hence when Ω is taken to be $\hat{G}$, our analysis on the square root of f and the space $L_{2,f}$ turns out to be important in prediction theory of such processes.

(b) In this paper our work is carried out under the separability assumption on $\mathscr{X}$. To justify this and the term "Banach'space"-valued processes as considered in [2–4], let $\mathscr{Y}$ be a separable Banach space and let $(\Omega, \mathfrak{B}, P)$ be a probability space. Let $\eta(\cdot)$ be a $\mathscr{Y}$-valued Gaussian random variable on $(\Omega, \mathfrak{B}, P)$, i.e., for each $y^* \in \mathscr{Y}^*$, the measurable function $y^*(\eta(\cdot))$ is a Gaussian random variable on $(\Omega, \mathfrak{B}, P)$. Obviously $y^*(\eta(\cdot))$ belongs to $L_2(\Omega, \mathfrak{B}, P)$. Then it is not hard to see [12] that the closed subspace $\mathfrak{S}\{y^*(\eta(\cdot)), y^* \in \mathscr{Y}^*\} \subseteq L_2(\Omega, \mathfrak{B}, P)$ is separable.

Let $\eta_n(\cdot)$, $-\infty < n < \infty$, be a $\mathscr{Y}$-valued process. We call it stationary if for any $y^* \in \mathscr{Y}^*$, the scalar-valued process $y^*(\eta_n(\cdot))$ is stationary. Define a new process ξ_n from $\mathscr{Y}^*$ into $\mathscr{K} = \mathfrak{S}\{y^*(\eta_n(\cdot)): y^* \in \mathscr{Y}^*, -\infty < n < +\infty\}$ by

$$\xi_n(y^*) = y^*(\eta_n(\cdot)).$$

Assuming that the random variables η_n are Gaussians it follows from the above discussion and the work in [12] that $\mathscr{K} = \mathfrak{S}\{\xi_n y^*, y^* \in \mathscr{Y}^*, -\infty < n < \infty\}$ is a separable closed subspace of $L_2(\Omega, \mathfrak{B}, P)$. Hence for the purpose of prediction theory it suffices to consider ξ_n from a separable closed subspace $\mathscr{X}$ of $\mathscr{Y}^*$ whose image $\bigcup_{-\infty<n<\infty} \xi_n(\mathscr{X})$ is dense in $\mathscr{K}$. It then follows that the spectral distribution and density of the process ξ_n can be considered as bounded linear operator on the Banach space $\mathscr{X}$ into its dual space $\mathscr{X}^*$. It is in this respect that our earlier factorization theorem [17] as well as the results of this paper, dealing with such spectral densities, lead to a fruitful study of the Banach-space-valued stationary stochastic processes.

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